第55卷 第6期 中山大学学报（自然科学版） Vol. 55 No. 6
2016年 11月 ACTA SCIENTIARUM NATURALIUM UNIVERSITATIS SUNYATSENI Nov. 2016

DOI:10.13471/j.cnki.acta.snus.2016.06.008

零度为 $n-4$ 的图的特征多项式*

吴廷增[1]，冯立华[2]，马海成[1]
（1. 青海民族大学数学与统计学院，青海 西宁 810007；
2. 中南大学数学系，湖南 长沙 410083）

摘　要： 图的零度是指图的邻接谱中零特征根的重数。显然，n 个顶点的图 G 的零度等于 n 减去其邻接矩阵的秩。计算了零度为 $n-4$ 的所有图的特征多项式。特别地，证明了许多零度为 $n-4$ 的图是谱唯一确定的，并构造了许多对零度为 $n-4$ 的同谱图。

关键词： 特征多项式；秩；零度；同谱图

中图分类号： O157.6；O151.26　　**文献标志码：** A　　**文章编号：** 0529-6579（2016）06-0057-07

On the characteristic polynomials of graphs with nullity $n-4$

WU Tingzeng[1], *FENG Lihua*[2], *MA Haicheng*[1]
(1. School of Mathematics and Statistics, Qinghai Nationalities University, Xining 810007, China;
2. Department of Mathematics, Central South University, Changsha 410083, China)

Abstract: The nullity of a graph is the multiplicity of zeroes in its adjacency spectrum. And the nullity of a graph G with n vertices equals to n minus the rank of adjacency matrix of G. The characteristic polynomials of graphs with nullity $n-4$ is computed. In particular, it is shown that some graphs with nullity $n-4$ are determined by their spectra. And some pairs of cospectral graphs with nullity $n-4$ are presented.

Key words: characteristic polynomial; rank; nullity; cospectral graphs

1 Introduction

Graphs considered in this paper are finite, undirected, and loopless. Undefined notation and terminology will follow [1]. Let $G = (V(G), E(G))$ be a graph with n vertices. For $S \subseteq V(G)$, the induced subgraph of G by S, denoted by $G[S]$, consists of $S$and all edges of G whose endpoints are contained in S. Let $G \cup H$ denote the union of two graphs G and H which have no common vertices. For any positive integer l, lG means the union of l disjoint copies of G. The path, cycle, star and complete graph of order n are denoted by P_n, C_n, $K_{1,n-1}$ and K_n, respectively. Let $c_i(G)$ denote the numbers of i-cycles in G. Let B be a set, and the number of elements in B is denoted by $|B|$.

The adjacency matrix of G is denoted by $A(G)$. The characteristic polynomial of a graph G, by definition, is

* **收稿日期：** 2015-11-28
基金项目： 国家自然基金资助项目（11101245，11271208，11301302，11561056）；青海省自然基金资助项目（2016-ZJ-947Q）；山东省自然基金资助项目（BS2013SF009）；中南大学数学与交叉科学项目资助项目；青海民族大学自然科学基金资助项目（2015XJZ12）
作者简介： 吴廷增（1978年生），男；**研究方向：** 数学化学、图的匹配理论和计算机网络；E-mail：mathtzwu@163.com

$$\varphi(G,x) = \det(xI - A(G)) = \sum_{k=0}^{n} a_k x^{n-k}$$

where I is the identity matrix of order n.

The *spectrum* of G consists of the eigenvalues together with their multiplicities of $A(G)$. Two graphs are said to be *cospectral* if they share the same spectrum or characteristic polynomial.

A graph is said to be *determined by the spectrum* (DS for short) if there is no other non-isomorphic graph with the same spectrum. The nullity of G, denoted by $\eta(G)$, is the multiplicity of zeroes in the spectrum of G. Let $r(G)$ be the rank of $A(G)$. Obviously, $\eta(G) = n - r(G)$. When $\eta(G) = 0$, the graph G is called nonsingular.

The problem characterizing all graphs G with $\eta(G) > 0$ is of great interest in both chemistry and mathematics. For a bipartite graph G which corresponds to an alterant hydrocarbon in chemistry, if $\eta(G) > 0$, it is indicated that the corresponding molecule is unstable. The nullity of a graph is also meaningful in mathematics since it is related to the singularity of adjacent matrix. The problem has not yet been solved completely. And it received a lot of attention from researchers in recent years, see [2 – 3]. Here we highlight two results which involve extremal nullity of graphs. That is, Cheng and Liu [4] characterized the graphs whose nullity reach the upper bounds $n-2$ and $n-3$. Chang et al. [5 – 6] masterly used the definition of multiplication of vertices (see p. 53 of [7]) to characterize all connected graphs with rank 4 or 5. More information see [8 – 9].

In this paper, we intend to compute the characteristic polynomials and spectra of graphs with nullity $n-4$. To fulfill our goal, we quote some definitions in [5]. Given a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$. A subset $M \subseteq V(G)$ is called an independent set of G if there are no edges between any two vertices in M. Let $\mathbf{m} = (m_1, m_2, \cdots m_n)$ be a vector of positive integers. Denote by $G \circ \mathbf{m}[m_1, m_2, \cdots, m_n]$ ($G \circ \mathbf{m}$ for short) the graph obtained from G by replacing each vertex v_i of G with an independent set of m_i vertices v_1^i, v_2^i, $\cdots$, $v_i^{m_i}$ and joining v_i^s with v_j^t if and only if v_i and v_j are adjacent in G. We say that $\{v_1^i, v_2^i, \cdots, v_i^{m_i}\}$ is the vertices of $G \circ \mathbf{m}$ corresponding to v_i. The resulting graph $G \circ \mathbf{m}$ is said to be obtained from G by multiplication of vertices. For graphs G_1, G_2, $\cdots$, G_k, we denote by $M(G_1, G_2, \cdots, G_k)$ the class of all graphs that can be obtained from one of the graphs in $\{G_1, G_2, \cdots, G_k\}$ by multiplication of vertices.

By the definition of $G \circ \mathbf{m}[m_1, m_2, \cdots, m_n]$, we can obtain some primary properties about the structure of $G \circ \mathbf{m}$ as follows.

Property 1 The number of vertices in $G \circ \mathbf{m}$ equals $\sum_{i=1}^{n} |m_i|$, where $|m_i| \geqslant 1$.

Property 2 Graph $G \circ \mathbf{m}$ contains just $|E(G)|$ complete bipartite induced subgraphs each of which is obtained from multiplying two vertices of an edge in G. Thus, the number of edges in $G \circ \mathbf{m}$ equals to $\sum_{e=uv \in E(G)} |m_u||m_v|$, where the sum is taken over all edges in G.

Property 3 For any triangle in $G \circ \mathbf{m}$, there exactly exists a complete 3-partite induced subgraph obtained by multiplying vertices on a triangle in G which contains it. Then

$$c_3(G \circ m) = \sum_{uvw} \binom{|m_u|}{1}\binom{|m_v|}{1}\binom{|m_w|}{1},$$

where uvw denotes a triangle in G, and the sum is taken over all triangles in G.

Property 4 Every 4-cycle in $G \circ \mathbf{m}$ must be contained in an induced subgraph obtained by multiplying every vertex on an edge, one P_3 or one C_4 in G. Direct computation yields

$$c_4(P_2 \circ m) = \binom{|m_u|}{2}\binom{|m_v|}{2}$$

where u and v are endpoints on P_2. Suppose that $P_3 = uvw$, and v is the vertex of degree two. There exist $\binom{|m_u|}{1}\binom{|m_v|}{2}\binom{|m_v|}{1}$ 4-cycles in $P_3 \circ \mathbf{m}$ each of which contains one vertex in m_u, two vertices in m_v and one vertex in m_w, respectively. Similarly, For a $C_4 = uvwz$, there exist $\binom{|m_u|}{1}\binom{|m_v|}{1}\binom{|m_w|}{1}\binom{|m_z|}{1}$ 4-cycles in $C_4 \circ \mathbf{m}$ each of which contains just one vertex in m_u, m_v, m_w and m_z. It follows that

$$c_4(G \circ m) = \sum_{uv = P_2 \in G} \binom{|m_u|}{2}\binom{|m_v|}{2} +$$

$$\sum_{u'v'w' = P_3 \in G} \binom{|m_{u'}|}{1}\binom{|m_{v'}|}{2}\binom{|m_{w'}|}{1} +$$

$$\sum_{u''v''w''z''=C_4\in G}\binom{|m_{u''}|}{1}\binom{|m_{v''}|}{1}\binom{|m_{w''}|}{1}\binom{|m_{z''}|}{1}$$

where the first sum is taken over all edges in G, the second sum is taken over all P_3's in G, and the third sum is taken over all C_4's in G.

This paper is organized as follows. In Section 2, we present some lemmas and characterize all graphs with nullity $n-4$. In Section 3, we compute the characteristic polynomials of graphs with nullity $n-4$. In Section 4, we investigate which graphs with nullity $n-4$ are DS. Precisely, we show that two classes of regular graphs and one class of non-bipartite graphs are DS. And we present some pairs of cospectral bipartite graphs.

2 Preliminaries

In this section, we will present some results which play a key role in the proofs of the main theorems.

Lemma 1[1] Let G be a graph with p vertices and q edges, and let $(d_1, d_2, \cdots, d_p)$ be the degree sequence of G. The coefficients of the characteristic polynomial of a graph G satisfy:

(**i**) $a_0 = 1$;

(**ii**) $a_1 = 0$;

(**iii**) $a_2 = -q$;

(**iv**) $a_3 = -2c_3(G)$;

(**v**) $a_4 = \binom{q}{2} - \sum_{i=1}^{p}\binom{d_i}{2} - 2c_4(G)$.

Lemma 2[10] Let G be a graph. For the adjacency matrix, the following can be obtained from the spectrum.

(**i**) The number of vertices.

(**ii**) The number of edges.

(**iii**) Whether G is regular.

(**iv**) Whether G is regular with any fixed girth.

(**v**) The number of closed walk of any length.

(**vi**) Whether G is bipartite.

Theorem 1[4] Suppose that G is a simple graph on n vertices and $n \geqslant 2$. Then $\eta(G) = n-2$ if and only if G is isomorphic to $K_{n_1,n_2} \cup kK_1$, where $n_1 + n_2 + k = n$, n_1 and $n_2 > 0$, and $k \geqslant 0$.

For $G \in M(G_1, G_2, \cdots, G_i)$, let $\widetilde{G}$ be the graph of order n obtained from G together with $n - |V(G)|$ isolated vertices. Furthermore, let

$\widetilde{M}(G_1, G_2, \cdots, G_i) = \{\widetilde{G}: G \in M(G_1, G_2, \cdots, G_i)\}$

Theorem 2[5] Let G be a connected graph. Then $r(G) = 4$ if and only if $G \in M(G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8)$, where $G_i (i = 1, 2, \cdots, 8)$ is depicted in Fig. 1.

Combining Theorems 1 and 2, we can obtain the following result.

Theorem 3 Let G be a simple graph on n vertices and $n \geqslant 4$. Then $\eta(G) = n-4$ if and only if $G \in \widetilde{M}(G_1, G_2, \cdots, G_9)$, where the graphs $G_1, \cdots, G_9$ are depicted in Fig. 1.

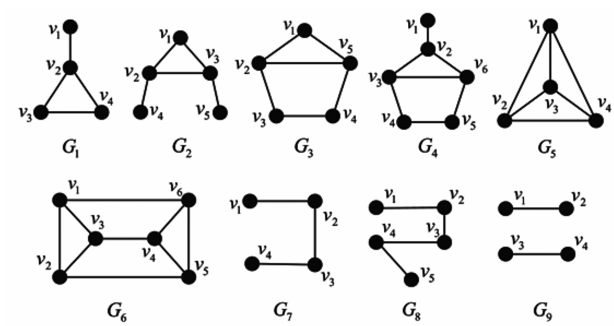

Fig. 1 The graphs $G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9$

3 The characteristic polynomials of graphs with nullity $n-4$

In this section, we will compute the characteristic polynomials of graphs with rank 4.

Theorem 4 Suppose that $G'_i = G_i \circ \mathbf{m}[m_1, \cdots, m_k]$, where G_i is depicted in Fig. 1, $i = 1, 2, \cdots, 9$, $k = |V(G_i) \leqslant 6|$, $|m_1| = a$, $|m_2| = b$, $|m_3| = c$, $|m_4| = d$, $|m_5| = e$ and $|m_6| = f$. Then each of the following holds.

(**i**) $\varphi(G'_1, x) = x^{a+b+c+d-4}[x^4 - (ab + bc + bd + cd)x^2 - 2bcdx + abcd]$.

(**ii**) $\varphi(G'_2, x) = x^{a+b+c+d+e-4}[x^4 - (ab + ac + bc + bd + ce)x^2 - 2abcx + abdc + abec + dbec]$.

(**iii**) $\varphi(G'_3, x) = x^{a+b+c+d+e-4}[x^4 - (ab + ae + be + bc + cd + de)x^2 - 2abex + abed + aecd + abcd + abec]$.

(**iv**) $\varphi(G'_4, x) = x^{a+b+c+d+e+f-4}[x^4 - (ab + bc + bf + cf + cd + de + ef)x^2 - 2bcfx + abed + abcd + bdec + abcf + abef + bcef + fbcd + fbed]$

(**v**) $\varphi(G'_5, x) = x^{a+b+c+d-4}[x^4 - (ab + ac + ad + bc + bd + cd)x^2 - 2(abc + abd + acd + bcd)x -$

$3abcd]$

(**vi**) $\varphi(G'_6,x) = x^{a+b+c+d+e+f-4}[x^4 - (ab + ac + af + bc + be + cd + df + de + fe)x^2 - 2(abc + def)x + abed + abdf + acbe + aced + acef + afed + abcd + fcde + abcf + bcef + fbcd + fbed]$

(**vii**) $\varphi(G'_7,x) = x^{a+b+c+d-4}[x^4 - (ab + bc + cd)x^2 + abcd]$.

(**viii**) $\varphi(G'_8,x) = x^{a+b+c+d+e-4}[x^4 - (ab + bc + cd + de)x^2 + abcd + bcde + abde]$.

(**ix**) $\varphi(G'_9,x) = x^{a+b+c+d-4}[x^4 - (ab + cd)x^2 + abcd]$.

Proof

(**i**) By Property 1, we note that G'_1 has $a + b + c + d$ vertices. Checking Fig. 1, we know that G_1 only has four edges, one triangle and five P_3's. By Properties 2 – 4, we obtain that G'_1 has $ab + bc + bd + cd$ edges, $\binom{b}{1}\binom{c}{1}\binom{d}{1}$ triangles and $\binom{a}{2}\binom{b}{2} + \binom{b}{2}\binom{c}{2} + \binom{b}{2}\binom{d}{2} + \binom{c}{2}\binom{d}{2} + \binom{a}{1}\binom{b}{2}\binom{c}{1} + \binom{a}{1}\binom{b}{2}\binom{c}{1} + \binom{b}{1}\binom{c}{2}\binom{d}{1} + \binom{c}{1}\binom{b}{2}\binom{d}{1} + \binom{c}{1}\binom{d}{2}\binom{b}{1}$ 4-cycles. By Lemma 1 and employing Maple 13.0 to perform the calculations, we obtain that

$$\varphi(G'_1,x) = x^{a+b+c+d} - (ab + bc + bd + cd)x^{a+b+c+d-2} - 2bcdx^{a+b+c+d-3} + abcdx^{a+b+c+d-4}$$

(**ii**) Similarly, by Property 1, we note that G'_2 has $a + b + c + d$vertices. Checking Fig. 1, it can be known that G'_2 has five edges, one triangle and seven P_3's. By Properties 2 – 4, we obtain that G'_2 has $ab + ac + bc + bd + ce$ edges, $\binom{a}{1}\binom{b}{1}\binom{c}{1}$ triangles and $\binom{a}{2}\binom{b}{2} + \binom{a}{2}\binom{c}{2} + \binom{b}{2}\binom{c}{2} + \binom{b}{2}\binom{d}{2} + \binom{c}{2}\binom{e}{2} + \binom{a}{1}\binom{b}{2}\binom{c}{1} + \binom{b}{1}\binom{a}{2}\binom{c}{1} + \binom{a}{1}\binom{c}{2}\binom{b}{1} + \binom{a}{1}\binom{b}{2}\binom{d}{1} + \binom{d}{1}\binom{b}{2}\binom{c}{1} + \binom{a}{1}\binom{c}{2}\binom{e}{1} + \binom{b}{1}\binom{c}{2}\binom{e}{1}$ 4-cycles. By Lemma 1, we obtain immediately that

$$\varphi(G'_2,x) = x^{a+b+c+d+e-4}[x^4 - (ab + ac + bc + bd + ce)x^2 - 2abcx + abdc + abec + dbec]$$

(**iii**) Checking Fig. 1, we note that G'_3 has six edges, one triangle, nine P_3's and one C_4. By Properties 1 – 4, there exists exactly $a + b + c + d$ vertices, $ab + ae + be + bc + cd + de$ edges, $\binom{a}{1}\binom{b}{1}\binom{e}{1}$ triangles and $\binom{a}{2}\binom{b}{2} + \binom{a}{2}\binom{e}{2} + \binom{b}{2}\binom{e}{2} + \binom{b}{2}\binom{c}{2} + \binom{c}{2}\binom{d}{2} + \binom{d}{2}\binom{e}{2} + \binom{a}{1}\binom{b}{2}\binom{e}{1} + \binom{a}{1}\binom{e}{2}\binom{b}{1} + \binom{b}{1}\binom{a}{2}\binom{e}{1} + \binom{a}{1}\binom{b}{2}\binom{c}{1} + \binom{b}{1}\binom{c}{2}\binom{d}{1} + \binom{c}{1}\binom{b}{2}\binom{e}{1} + \binom{c}{1}\binom{d}{2}\binom{e}{1} + \binom{d}{1}\binom{e}{2}\binom{b}{1} + \binom{a}{1}\binom{e}{2}\binom{d}{1} + \binom{b}{1}\binom{c}{1}\binom{d}{1}\binom{e}{1}$ 4-cycles in G'_3. By Lemma 1, we have

$$\varphi(G'_3,x) = x^{a+b+c+d+e-4}[x^4 - (ab + ae + be + bc + cd + de)\cdot x^2 - 2abex + abed + adec + abcd + abec]$$

(**iv**) Checking Fig. 1, we again note that G_4 has seven P_2's, one triangle, eleven P_3's and one C_4. By Properties 1 – 4, it is easy to obtain that G'_4 has $a + b + c + d + e + f$ vertices, $ab + bc + bf + cd + de + ef$ edges, $\binom{b}{1}\binom{c}{1}\binom{f}{1}$ triangles and $\binom{a}{2}\binom{b}{2} + \binom{b}{2}\binom{f}{2} + \binom{b}{2}\binom{c}{2} + \binom{c}{2}\binom{f}{2} + \binom{d}{2}\binom{e}{2} + \binom{e}{2}\binom{f}{2} + \binom{c}{2}\binom{d}{2} + \binom{a}{1}\binom{b}{2}\binom{c}{1} + \binom{a}{1}\binom{b}{2}\binom{f}{1} + \binom{c}{1}\binom{b}{2}\binom{f}{1} + \binom{b}{1}\binom{c}{2}\binom{f}{1} + \binom{b}{1}\binom{f}{2}\binom{c}{1} + \binom{b}{1}\binom{c}{2}\binom{d}{1} + \binom{d}{1}\binom{c}{2}\binom{f}{1} + \binom{c}{1}\binom{d}{2}\binom{e}{1} + \binom{d}{1}\binom{e}{2}\binom{f}{1} + \binom{e}{1}\binom{f}{2}\binom{c}{1} + \binom{e}{1}\binom{f}{2}\binom{b}{1}$ 4-cycles. By Lemma 1, we get that

$$\varphi(G'_4,x) = x^{a+b+c+d+e+f-4}[x^4 - (ab + bc + bf + cf + cd + de + ef)x^2 - 2bcfx + abed + abcd + bdec + abcf + abef + bcef + fbcd + fbed]$$

(**v**) Observing Fig. 1, we see that G_5 has six P_2's, four triangles, twelve P_3's and three C_4's. By Properties 1 – 4, G'_5 has $a + b + c + d$ vertices, $ab + ac + ad + bc + bd + cd$ edges, $\binom{a}{1}\binom{b}{1}\binom{c}{1} + \binom{a}{1}\binom{b}{1}\binom{d}{1} + \binom{b}{1}\binom{c}{2}\binom{d}{1} + \binom{a}{1}\binom{c}{1}\binom{d}{1}$ triangles and $\binom{a}{2}\binom{b}{2} + \binom{a}{2}\binom{c}{2} + \binom{a}{2}\binom{d}{2} + \binom{b}{2}\binom{c}{2} + \binom{b}{2}\binom{d}{2} + \binom{c}{2}\binom{d}{2} + \binom{b}{1}\binom{a}{2}\binom{c}{1} + \binom{b}{1}\binom{a}{2}\binom{d}{1} + \binom{c}{1}\binom{a}{2}\binom{d}{1} + \binom{a}{1}\binom{b}{2}\binom{c}{1} + \binom{a}{1}\binom{b}{2}\binom{d}{1} + \binom{c}{1}\binom{b}{2}\binom{d}{1} + \binom{b}{1}\binom{c}{2}\binom{a}{1} + \binom{b}{1}\binom{c}{2}\binom{d}{1} +$

$\binom{a}{1}\binom{c}{2}\binom{d}{1}+\binom{c}{1}\binom{d}{2}\binom{a}{1}+\binom{c}{1}\binom{d}{2}\binom{b}{1}+\binom{b}{1}\binom{d}{2}\binom{a}{1}+3abcd$ 4-cycles. By Lemma 1, we have

$$\varphi(G'_5,x)=x^{a+b+c+d-4}[x^4-(ab+ac+ad+bc+bd+cd)x^2-2(abc+abd+acd+bcd)x-3abcd]$$

(**vi**) Observing Fig. 1, it is not difficult to know that G_6 has nine edges, two triangles, eighteen P_3's and three C_4's. These mean, by Properties 1 – 4, that G'_6 has $a+b+c+d+e+f$ vertices, $ab+ac+af+bc+bd+cd+de+df+ef$ edges, $\binom{a}{1}\binom{b}{1}\binom{c}{1}+\binom{d}{1}\binom{e}{1}\binom{f}{1}$ triangles and $\binom{a}{2}\binom{b}{2}+\binom{a}{2}\binom{c}{2}+\binom{a}{2}\binom{f}{2}+\binom{b}{2}\binom{c}{2}+\binom{b}{2}\binom{e}{2}+\binom{c}{2}\binom{d}{2}+\binom{d}{2}\binom{e}{2}+\binom{d}{2}\binom{f}{2}+\binom{e}{2}\binom{f}{2}+\binom{b}{1}\binom{a}{2}\binom{c}{1}+\binom{b}{1}\binom{a}{2}\binom{f}{1}+\binom{c}{1}\binom{a}{2}\binom{f}{1}+\binom{a}{1}\binom{b}{2}\binom{c}{1}+\binom{a}{1}\binom{b}{2}\binom{e}{1}+\binom{c}{1}\binom{b}{2}\binom{e}{1}+\binom{a}{1}\binom{c}{2}\binom{b}{1}+\binom{a}{1}\binom{c}{2}\binom{d}{1}+\binom{b}{1}\binom{c}{2}\binom{d}{1}+\binom{c}{1}\binom{d}{2}\binom{e}{1}+\binom{c}{1}\binom{d}{2}\binom{f}{1}+\binom{f}{1}\binom{d}{2}\binom{e}{1}+\binom{b}{1}\binom{e}{2}\binom{c}{1}+\binom{b}{1}\binom{e}{2}\binom{f}{1}+\binom{d}{1}\binom{e}{2}\binom{f}{1}+\binom{a}{1}\binom{f}{2}\binom{e}{1}+\binom{a}{1}\binom{f}{2}\binom{d}{1}+\binom{d}{1}\binom{f}{2}\binom{e}{1}+abef+acdf+bcde$ 4-cycles. By Lemma 1, we have

$$\varphi(G'_6,x)=x^{a+b+c+d+e+f-4}\cdot[x^4-(ab+ac+af+bc+be+cd+df+de+fe)x^2-2(abc+def)x+abed+abdf+acbe+aced+acef+afed+abcd+fcde+abcf+bcef+fbcd+fbed]$$

(**vii**) Observing Fig. 1, we note that there exist three P_2's and two P_3's in G_7. These imply, by Properties 1, 2 and 4, that G'_7 has $a+b+c+d$ vertices, $ab+bc+cd$ edges and $\binom{a}{2}\binom{b}{2}+\binom{b}{2}\binom{c}{2}+\binom{c}{2}\binom{d}{2}+\binom{a}{1}\binom{b}{2}\binom{c}{1}+\binom{b}{1}\binom{c}{2}\binom{d}{1}$ 4-cycles. By Lemma 1, we obtain that

$$\varphi(G'_7,x)=x^{a+b+c+d-4}\cdot[x^4-(ab+bc+cd)x^2+abcd]$$

(**viii**) Observing Fig. 1, we see that G_8 has four P_2's and three P_3's. By Properties 1, 2 and 4, it can be seen that G'_8 has $a+b+c+d$ vertices, $ab+bc+cd$ edges and $\binom{a}{2}\binom{b}{2}+\binom{b}{2}\binom{c}{2}+\binom{c}{2}\binom{d}{2}+\binom{d}{2}\binom{e}{2}+\binom{a}{1}\binom{b}{2}\binom{c}{1}+\binom{b}{1}\binom{c}{2}\binom{d}{1}+\binom{c}{1}\binom{d}{2}\binom{e}{1}$ 4-cycles. By Lemma 1, we obtain that

$$\varphi(G'_8,x)=x^{a+b+c+d+e-4}[x^4-(ab+bc+cd+ed)x^2+abcd]$$

(**ix**) Checking Fig. 1, we note that G_9 has two edges. By Properties 1, 2 and 4, we know that G'_9 has $a+b+c+d$ vertices, $ab+cd$ edges and $\binom{a}{2}\binom{b}{2}+\binom{c}{2}\binom{d}{2}$ 4-cycles. By Lemma 1, we have

$$\varphi(G'_9,x)=x^{a+b+c+d-4}\cdot[x^4-(ab+cd)x^2+abcd]$$

By the above argument, we obtain the desired result.

Remark 1　By Theorem 4, we may compute the spectrum of a graph $G\in\widetilde{M}(G_1,G_2,\cdots,G_9)$.

4 The spectral characterization of graphs with nullity $n-4$

Which graphs are DS? This problem is a long standing question in Spectral Graph Theory. van Dam and Haemers [10 – 11] gave two excellent surveys regarding the development of this question. In particular, the usual adjacency matrix was addressed. Recently, many new results are found, see [12 – 14]. In this section, we will investigate which graph $G\in\widetilde{M}(G_1,G_2,\cdots,G_9)$ is DS.

Ma and Ren [15] investigated which kind of complete multipartite graph is DS since it is well known that not all complete multipartite graphs are DS. And they proved the following results.

Theorem 5[15]　Let $p_1,p_2,\cdots p_s$ be the prime numbers and t be a non-negative integer. Then $K_{\underbrace{1,\cdots,1}_{t},p_1,p_2\cdots p_s}$ is DS.

Theorem 6[15]　Let d,s be two positive integers. Then $K_{\underbrace{d+1,\cdots,d+1}_{r},\underbrace{d,\cdots,d}_{s-r}}$ is DS, where $0\leqslant r<s$.

By Theorems 5 and 6, we obtain directly a corollary as follows.

Theorem 7

(**i**) If $s+t=4$, then $K_{\underbrace{1,\cdots,1}_{t},p_1,p_2,\cdots,p_s}$ is DS.

(**ii**) The complete regular 4-partite graph is DS.

Additionally, by the definition of $G_5 \circ \mathbf{m}[m_1, m_2, m_3, m_4]$, we note that $G_5 \circ \mathbf{m}$ is a complete 4-partite graphs. Observing Theorem 4, it is easy to see that only the fifth coefficient of characteristic polynomial of $G_5 \circ \mathbf{m}$ is negative, which implies that the following proposition.

Proposition 1 Let $G \in \widetilde{M}(G_5)$. For any graph H, if H and G are cosperctal, then $H \in \widetilde{M}(G_5)$.

The result of Proposition 1 indicates that if no two graphs in $\widetilde{M}(G_5)$ are cospectral, then every graph in $\widetilde{M}(G_5)$ is DS.

Theorem 8 Suppose that $G = G_6 \circ \mathbf{m}[m_1, m_2, m_3, m_4, m_5, m_6]$ and $|m_i| = s$ for $i = 1, \cdots, 6$. Then G is DS.

Proof. By the definition of multiplication of vertices, there exist exactly three graphs $G_5 \circ \mathbf{m}[m_1, m_2, m_3, m_4]$ (here $|m_1| = |m_2| = |m_3| = |m_4| = r$), $G_6 \circ \mathbf{m}[m_1, m_2, m_3, m_4, m_5, m_6]$ (here $|m_1| = |m_2| = |m_3| = |m_4| = |m_5| = |m_6| = s$) and $G_9 \circ \mathbf{m}[m_1, m_2, m_3, m_4]$ (here $|m_1| = |m_2| = |m_3| = |m_4| = t$) are regular in $\widetilde{M}(G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9)$. By Proposition 1 and Lemma 2 (iii) and (vi), G is DS.

By the definition of $G \circ \mathbf{m}$, we know that $2K_{t,t}$ is isomorphic to $G_9 \circ \mathbf{m}[m_1, m_2, m_3, m_4]$ when $|m_1| = |m_2| = |m_3| = |m_4| = t$. Hence, from the proof of Theorem 8, we easily obtain the following result.

Theorem 9 $2K_{t,t}$ is DS.

Theorem 10 Let $G = G'_1 \cup rK_1$, where $b = c = d = 1$ in G'_1. Then G is DS.

Proof. Suppose that G has $a' + r + 3$ vertices. Checking G, we note that it only contains one triangle. This implies, by Lemma 2 (v), that if graph H is cospectral with G, then H must contain one triangle. By Theorems 3 and 4, $G'_2 \cup sK_1$ (here $a = b = c = 1$ in G'_3) and G'_4 (here $b = c = f = 1$ in G'_4) contains one triangle, respectively. In the following we consider three cases.

Case 1. Assume that G and $G'_2 \cup sK_1$ are cospectral. Therefore, G and $G'_2 \cup sK_1$ have the same vertices and coefficients of their characteristic polynomials. By Theorem 4 (i) and (ii), we have

$$\begin{cases} a' + r + 3 = 3 + d + e + s, \\ a' + 3 = 3 + d + e, \\ a' = d + e + de. \end{cases}$$

Solving this equation system, we obtain that $d = 0$ or $e = 0$. This contradicts the fact $d, e > 0$. Hence G and $G'_2 \cup sK_1$ are not cospectral.

Case 2. Suppose that G and $G'_3 \cup tK_1$ are cospectral. Then G and $G'_3 \cup tK_1$ have the same vertices and coefficients of their characteristic polynomials. By Theorem 4 (i) and (iii), we obtain that

$$\begin{cases} a' + r + 3 = 3 + c + d + t, \\ a' + 3 = 3 + c + d + cd, \\ a' = c + d + 2cd \end{cases}$$

Solving this equation system, we get that $c = 0$ or $d = 0$. This contradicts the fact $c, d > 0$. Hence G and $G'_3 \cup tK_1$ are not cospectral.

Case 3. We assume that G and $G'_4 \cup gK_1$ are cospectral. Then G and $G'_4 \cup gK_1$ have the same vertices and coefficients of their characteristic polynomials. By Theorem 4 (i) and (iv), we get that

$$\begin{cases} a' + r + 3 = 3 + a + d + e + g, \\ a' + 3 = 3 + a + d + e + de, \\ a' = a + d + e + ad + ae + 3de \end{cases}$$

Solving the equation system above, we obtain that $ad + ae + 2de = 0$. However, by the definition of G'_4, one has $a, d, e > o$, which implies that $ad + ae + 2de > 0$, a contradiction. Thus, G and $G'_4 \cup gK_1$ are not cospectral.

From the argument above, we obtain that G is DS.

In fact, to characterize DS graphs with nullity $n - 4$ is a very hard problem. There exist some graphs in $\widetilde{M}(G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9)$ are cospectral. For example, $K_{1,4}$ and $C_4 \cup K_1$ are cospectral. Furthermore, we have the following results.

Proposition 2 Let $G = G_9 \circ \mathbf{m}[m_1, m_2, m_3, m_4]$ with $|m_1| = 1$, $|m_2 = 4|$, $|m_3| = s$ and $|m_4| = t$, and let $H = G_9 \circ \mathbf{m}[m_1, m_2, m_3, m_4]$ with $|m_1| = 2$ and $|m_2| = 2$, $|m_3| = s$ and $|m_4| = t$. Then $G \cup kK_1$ and $H \cup (k+1)K_1$ are cospectral.

Proposition 3 Let $G = G_7 \circ \mathbf{m}[m_1, m_2, m_3, m_4]$ with $|m_1| = a + 3$, $|m_2 = 1|$, $|m_3| = 1$ and $|m_4| = 2a + 3$, and let $H = G_8 \circ \mathbf{m}[m_1, m_2, m_3, m_4. m_5]$ with $|m_1| = a + 1$, $|m_2| = 1$, $|m_3| = 1$, $|m_4| = 1$ and $|m_5| = 2a + 4$, where $a \geqslant 0$. Then $G \cup kK_1$ and $H \cup kK_1$ are cospectral, where $k \geqslant 0$.

Proof. By Theorem 4 (vii) and (viii), we obtain that

$$\varphi(G,x)=x^{3a+k+4}\cdot[x^4-(3a+7)x^2+3a^2+9a+9]=\varphi(H,x)$$

Therefore, $G\cup kK_1$ and $H\cup kK_1$ are cospectral, where $a,k\geqslant 0$.

5 Conclusion

In this paper, we computed the characteristic polynomials of graphs with rank 4. Analogously, we may compute the characteristic polynomials of graphs with rank 5 (These graphs are characterized by Chang et al [6]). But the process "would be very cumbersome". At present time, we cannot find a unified approach to answer the problem "which graph $G\in\widetilde{M}(G_1,G_2,G_3,G_4,G_5,G_6,G_7,G_8,G_9)$ is DS?". We only showed that few families of graphs with nullity $n-4$ are DS. Hence we expect to find more graph classes with nullity $n-4$ each of which is DS. In particular, it would be an interesting question to characterize a graph in $\widetilde{M}(G_5)$ is DS.

References:

[1] BIGGS N. Algebraic graph theory [M]. Cambridge: Cambridge University Press, 1993.

[2] BOROVICANIN B, GUTMAN I. Nullity of graphs [C]. Applications of Graph Spectra, Math Inst. Belgrade, 2009:107 - 122.

[3] GUTMAN I, BOROVICANIN B. Nullity of graphs: an updated survey [C]. Selected Topics on Applications of Graph Spectra, Math Inst. Belgrade, 2011: 137 - 154.

[4] CHENG B, LIU B. On the nullity of graphs [J]. Electron J Linear Algebra Ela, 2007, 16: 60 - 67.

[5] CHANG G, HUANG L, YEH H. A characterization of graphs with rank 4 [J]. Linear Algebra Appl, 2011, 434(8): 1793 - 1798.

[6] CHANG G, HUANG L, YEH H. A characterization of graphs with rank 5 [J]. Linear Algebra Appl, 2012, 436(11): 4241 - 4250.

[7] GOLUMBIC M. Algorithmic graph theory and perfect graphs [M]. Amsterdam: North-Holland Publishing Co, 2004.

[8] WANG X, ZHAO X, YAO B. Spanning trees of totally edge-growing network models [J]. Acta Scientiarum Naturalium Universitatis Sunyatseni, 2016, 55(1): 48 - 53.

[9] TANG B, REN H. The number of perfect matching in two types of 3-regular graph [J]. Acta Scientiarum Naturalium Universitatis Sunyatseni, 2014, 53 (5): 54 - 58.

[10] van DAM E, HAEMERS W. Which graphs are determined by their spectrum? [J]. Linear Algebra Appl, 2003, 373: 241 - 272.

[11] van DAM E, HAEMERS W. Developments on spectral characterizations of graphs [J]. Discrete Math, 2009, 309: 576 - 586.

[12] CAMARA M, HAEMERS W. Spectral characterizations of almost complete graphs [J]. Discrete Appl Math, 2014, 176: 19 - 23.

[13] LIU F, HUANG Q, LAI H. Note on the spectral characterization of some cubic graphs with maximum number of triangles [J]. Linear Algebra Appl, 2013, 438(3): 1393 - 1397.

[14] ZHANG X, ZHANG H. Some graphs determined by their spectra [J]. Linear Algebra Appl, 2009, 431 (9): 1443 - 1454.

[15] MA H, REN H. On the spectral characterization of the union of complete multipartite graph and some isolated vertices [J]. Discrete Math, 2010, 310: 3648 - 3652.